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DAMPED SOLUTIONS OF SINGULAR IVPS WITH ϕ-LAPLACIAN

41.1 INTRODUCTION

We study the equation

$$(p(t)\phi(u'(t)))' + p(t)f(\phi(u(t))) = 0,\ t > 0 \tag{41.1}$$

with the initial conditions

$$u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, L], \tag{41.2}$$

where

$$\phi \in C^1(\mathbb{R}), \quad \phi'(x) > 0 \text{ for } x \in (\mathbb{R} \setminus \{0\}), \tag{41.3}$$

$$\phi(\mathbb{R}) = \mathbb{R}, \quad \phi(0) = 0, \tag{41.4}$$

$$L_0 < 0 < L, \quad f(\phi(L_0)) = f(0) = f(\phi(L)) = 0, \tag{41.5}$$

$$f \in C[\phi(L_0), \phi(L)], \quad xf(x) > 0 \text{ for } x \in ((\phi(L_0), \phi(L)) \setminus \{0\}), \tag{41.6}$$

$$p \in C[0,\infty) \cap C^1(0,\infty), \quad p'(t) > 0 \text{ for } t \in (0,\infty), \quad p(0) = 0. \tag{41.7}$$

Remark 41.1. Let us note that the integral $\int_0^1 \frac{\mathrm{d}s}{p(s)}$ may be divergent and, due to the assumption $p(0) = 0$, the equation (41.1) has a singularity at $t = 0$.

Example 41.2. Consider $\phi(x) = |x|^\alpha \operatorname{sgn} x$, $x \in \mathbb{R}$, where $\alpha \geq 1$. Then $\phi'(x) = \alpha|x|^{\alpha-1}$ and conditions (41.3) and (41.4) are fulfilled. As an example of f satisfying conditions (41.5) and (41.6) we can take $f(x) = x\,(x - \phi(L_0))\,(\phi(L) - x)$, $x \in \mathbb{R}$. If we take $p(t) = t^\beta$, $t \in [0, \infty)$, where $\beta > 0$, then p fulfils (41.7).

We have been motivated by real world problems from hydrodynamics, nonlinear field theory, population genetics, homogeneous nucleation theory or relativistic cosmology.

The following system of PDE's was derived to study the behaviour of non-homogenous fluids in the Cahn-Hilliard theory used in hydrodynamics

$$\rho_t + \operatorname{div}(\rho v) = 0, \quad \frac{\mathrm{d}v}{\mathrm{d}t} + \nabla(\mu(\rho) - \gamma\Delta\rho) = 0,$$

where ρ is the density, v the velocity of the fluid, $\mu(\rho)$ is its chemical potential and γ is constant. If the motion of the fluid is zero, the state of the fluid in $\mathbb{R}^N$ is described by the equation

$$\gamma\Delta\rho = \mu(\rho) - \mu_0,$$

where γ and μ_0 are suitable constants. When we search for a solution with the spherical symmetry, the equation is reduced to the ordinary differential equation

$$\gamma\left(\rho'' + \frac{N-1}{r}\rho'\right) = \mu(\rho) - \mu_0, \quad r \in (0,\infty).$$

Together with the boundary conditions

$$\rho'(0) = 0, \quad \lim_{r\to\infty} \rho(r) = \rho_\ell > 0,$$

it describes the formation of microscopic bubbles in a fluid, in particular, vapor inside liquid. The condition $\rho'(0) = 0$ follows from the central symmetry and it is necessary for the smoothness of solution of the singular equation at $r = 0$ and the condition $\lim_{r\to\infty} \rho(r) = \rho_\ell$ means that the bubble is surrounded by an external liquid with density ρ_ℓ.

Let $N = 3$. In the simplest model for a non-homogeneous fluid, the problem is reduced to the form

$$\left(t^2u'\right)' = 4\lambda^2 t^2 (u+1)u(u-\xi), \tag{41.8}$$

$$u(\infty) = \xi, \quad u'(0) = 0,$$

where $\lambda \in (0,\infty)$ and $\xi \in (0,1)$ are parameters. Many important physical properties of the bubbles depend on the existence of an increasing solution of the problem with just one zero. In particular, the gas density inside the bubble, the bubble radius and the surface tension.

Equation (41.8) is a special case of our equation (41.1).

The equation

$$(p(t)u'(t))' + q(t)f(u(t)) = 0,$$

without ϕ-Laplacian, was investigated for $p \equiv q$ in [7–10], for $p \not\equiv q$ in [1, 3, 11, 12] and with ϕ-Laplacian in [2, 4–6].

Definition 41.3. A function $u \in C^1[0,\infty)$ with $\phi(u') \in C^1(0,\infty)$ which satisfies equation (41.1) for every $t \in (0,\infty)$ is called a *solution* of equation (41.1). If moreover u satisfies the initial conditions (41.2), then u is called a *solution* of problem (41.1), (41.2).

Definition 41.4. Consider a solution u of problem (41.1), (41.2) with $u_0 \in (L_0, L)$ and denote

$$u_{sup} = \sup\{u(t)\colon t \in [0,\infty)\}.$$

If $u_{sup} < L$, then u is called a *damped solution* of problem (41.1), (41.2).
If $u_{sup} = L$, then u is called a *homoclinic solution* of problem (41.1), (41.2).
The homoclinic solution is called a *regular homoclinic solution*, if $u(t) < L$ for $t \in [0,\infty)$ and a *singular homoclinic solution*, if there exists $t_0 > 0$ such that $u(t_0) = L$.
If $u_{sup} > L$, then u is called an *escape solution* of problem (41.1), (41.2).

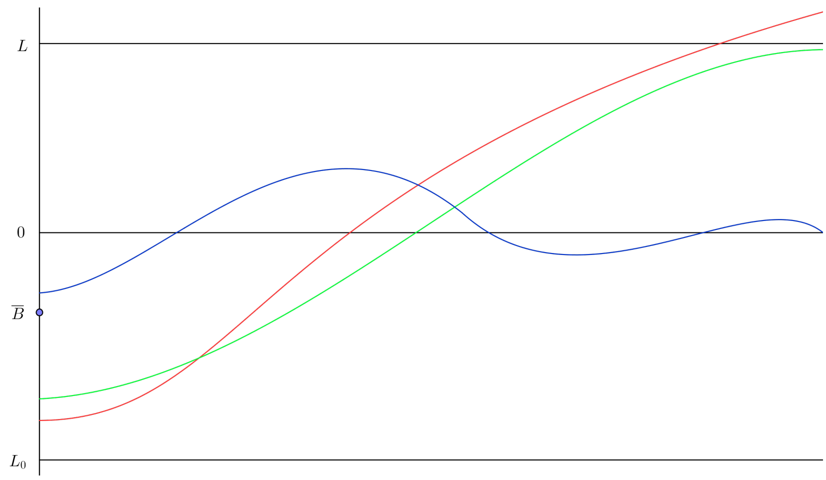

Fig. 41.1 Three types of solutions.

Source: own elaboration

Remark 41.5. Equation (41.1) has the constant solutions $u(t) \equiv L$, $u(t) \equiv 0$ and $u(t) \equiv L_0$.

41.2 EXISTENCE OF SOLUTION OF AUXILIARY PROBLEM

To prove the existence of damped solution, we define an auxiliary equation with a bounded nonlinearity.

The auxiliary equation has the form

$$(p(t)\phi(u'(t)))' + p(t)\tilde{f}(\phi(u(t))) = 0, \quad t \in (0,\infty), \tag{41.9}$$

where

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in [\phi(L_0), \phi(L)], \\ 0 & \text{for } x < \phi(L_0),\ x > \phi(L). \end{cases} \tag{41.10}$$

Theorem 41.6 (Existence of solutions of problem (41.9), (41.2), [2])**.** *Assume* (41.3)–(41.7)*. Then, for each* $u_0 \in [L_0, L]$*, there exists a solution* u *of problem* (41.9), (41.2).

Example 41.7. Consider $\phi\colon \mathbb{R} \to \mathbb{R}$ given by one of the next formulas

$$\phi(x) = |x|^\alpha \operatorname{sgn} x, \quad \alpha > 1, \tag{41.11}$$

$$\phi(x) = \left(x^4 + 2x^2\right) \operatorname{sgn} x, \tag{41.12}$$

$$\phi(x) = \sinh x = \frac{e^x - e^{-x}}{2}\,, \tag{41.13}$$

$$\phi(x) = \operatorname{arg\,sinh} x = \ln\left(x + \sqrt{x^2+1}\right), \tag{41.14}$$

$$\phi(x) = \ln(|x|+1) \operatorname{sgn} x. \tag{41.15}$$

Assume that $\phi(L) < -\phi(L_0)$ and put

$$p(t) = t^\beta, \quad t \in [0,\infty)\,,\ \beta > 0,$$
$$f(x) = k|x|^\gamma \operatorname{sgn} x(x - \phi(L_0))(\phi(L) - x), \quad x \in [\phi(L_0), \phi(L)],\ \gamma > 0,\ k > 0.$$

Then the functions p, ϕ and f fulfil all assumptions of Theorem 41.6. In particular $\phi \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ for each ϕ given by (41.11)–(41.15). Therefore the auxiliary problem (41.9), (41.2) has a solution for every $u_0 \in [L_0, L]$.

41.3 UNIQUENESS AND CONTINUOUS DEPENDENCE ON INITIAL VALUES FOR AUXILIARY PROBLEM

The presence of ϕ-Laplacian in equation (41.1) brings difficulties in the study of the uniqueness. For example, if $\phi(x) = |x|^\alpha \operatorname{sgn} x$ and $\alpha > 1$, then ϕ fulfils the Lipschitz condition on $\mathbb{R}$. On the other hand, $\phi^{-1} = |x|^{\frac{1}{\alpha}} \operatorname{sgn} x$ and $\left(\phi^{-1}\right)'(x) = \frac{1}{\alpha}|x|^{\frac{1}{\alpha}-1}$. Thus we get $\lim_{x\to 0}\left(\phi^{-1}\right)'(x) = \infty$ and the function ϕ^{-1} does not fulfil the Lipschitz condition in the neighbourhood of zero. Since both ϕ and ϕ^{-1} must be present in the operator form of problem (41.1), (41.2), we cannot use the standard approach with a Lipschitz constant to prove the uniqueness near zero. Therefore, we distinguish two cases:

- The first case, where functions ϕ^{-1} and f are Lipschitz continuous. The uniqueness of solution of problem (41.1), (41.2) is guaranteed.
- The second case, where function ϕ^{-1} do not have to be Lipschitz continuous.

Theorem 41.8 (Uniqueness and continuous dependence on initial values I, [2])**.** *Assume* (41.3)–(41.7) *and*

$$f \in \mathrm{Lip}\left[\phi(L_0), \phi(L)\right], \tag{41.16}$$

$$\phi^{-1} \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}). \tag{41.17}$$

Let u_i *be a solution of problem* (41.9), (41.2) *with* $u_0 = B_i \in [L_0, L]$, $i = 1, 2$. *Then, for each* $\beta > 0$, *there exists* $K > 0$ *such that*

$$\|u_1 - u_2\|_{C^1[0,\beta]} \le K|B_1 - B_2|.$$

Furthermore, any solution of problem (41.9), (41.2) *with* $u_0 \in [L_0, L]$ *is unique on* $[0,\infty)$.

Example 41.9. We need both ϕ and ϕ^{-1} from $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ in order to apply Theorem 41.8. Let us check the functions ϕ in Example 41.7

$$\begin{array}{lll}
\phi(x) = |x|^{\alpha}\operatorname{sgn} x, \quad \alpha > 1 & \Rightarrow \phi^{-1}(x) = |x|^{\frac{1}{\alpha}}\operatorname{sgn} x & \notin \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}),\\
\phi(x) = \left(x^4 + 2x^2\right)\operatorname{sgn} x, & \Rightarrow \phi^{-1}(x) = \sqrt{\sqrt{|x|+1} - 1} & \notin \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}),\\
\phi(x) = \sinh x = \dfrac{e^x - e^{-x}}{2}, & \Rightarrow \phi^{-1}(x) = \operatorname{arg\,sinh} x & \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}),\\
\phi(x) = \operatorname{arg\,sinh} x = \ln\left(x + \sqrt{x^2+1}\right) & \Rightarrow \phi^{-1}(x) = \sinh x & \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}),\\
\phi(x) = \ln(|x|+1)\operatorname{sgn} x & \Rightarrow \phi^{-1}(x) = \left(e^{|x|-1}\right)\operatorname{sgn} x & \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}).
\end{array}$$

Consider p, f from Example 41.7 with $\gamma \geq 1$ and ϕ given by one of the formulas (41.13)–(41.15). Then all assumptions of Theorem 41.8 are fulfilled and problem (41.9), (41.2) has a unique solution for $u_0 \in [L_0, L]$. Note that if $\gamma \in (0,1)$, then f is not Lipschitz continuous on a neighbourhood of zero, that is (41.16) is not valid. Similarly, in the case that ϕ is given by (41.11) or (41.12), then ϕ^{-1} is not Lipschitz continuous on a neighbourhood of zero and hence (41.17) falls.

In next two theorems, we show assumptions under which solutions of problem (41.9), (41.2) continuously depend on their initial values in the case that ϕ^{-1} is not locally Lipschitz continuous.

To derive further important properties of solutions of (41.9), we need to assume

$$\exists \bar{B} \in (L_0, 0)\colon \tilde{F}\left(\bar{B}\right) = \tilde{F}(L), \qquad \text{where } \tilde{F}(x) = \int_0^x \tilde{f}(\phi(s))\mathrm{d}s, \quad x \in \mathbb{R} \tag{41.18}$$

and

$$\limsup_{t\to\infty} \frac{p'(t)}{p(t)} < \infty. \tag{41.19}$$

Theorem 41.10 (Continuous dependence on initial values II, [2])**.** *Assume* (41.3)–(41.7), (41.16), (41.18), (41.19) *and*

$$\limsup_{x\to 0^-} \left(-x\left(\phi^{-1}\right)'(x)\right) < \infty, \quad \phi' \textit{ is nonincreasing on } (-\infty, 0). \tag{41.20}$$

Let B_1, B_2 satisfy

$$B_1 \in (2\varepsilon, L - 2\varepsilon), \quad |B_1 - B_2| < \varepsilon$$

for some $\varepsilon > 0$. Let u_i be a solution of problem (41.9), (41.2) *with $u_0 = B_i$, $i = 1, 2$. Then for each $\beta > 0$ where*

$$u_i' < 0 \textit{ on } (0, \beta], \; i = 1, 2,$$

there exists $K \in (0, \infty)$ such that

$$\|u_1 - u_2\|_{C^1[0,\beta]} \leq K|B_1 - B_2|.$$

Theorem 41.11 (Continuous dependence on initial values III, [2]). *Assume* (41.3)–(41.7), (41.16), (41.18), (41.19) *and*

$$\limsup_{x\to 0^+}\left(x\left(\phi^{-1}\right)'(x)\right)<\infty, \quad \phi' \text{ is nondecreasing on } (0,\infty). \tag{41.21}$$

Let B_1, B_2 *satisfy*

$$B_1\in(L_0+2\varepsilon,-2\varepsilon)\,,\quad |B_1-B_2|<\varepsilon$$

for some $\varepsilon>0$. *Let* u_i *be a solution of problem* (41.9), (41.2) *with* $u_0=B_i$, $i=1,2$. *Then for each* $\beta>0$ *where*

$$u_i'>0 \text{ on } (0,\beta],\ i=1,2,$$

there exists $K\in(0,\infty)$ *such that*

$$\|u_1-u_2\|_{C^1[0,\beta]}\le K|B_1-B_2|.$$

41.4 EXISTENCE AND UNIQUENESS OF DAMPED SOLUTION

Lemma 41.12 (Estimates, [2]). *Let assumptions* (41.3)–(41.7), (41.18) *and* (41.19) *hold. Let* u *be a solution of problem* (41.9), (41.2) *with* $u_0\in(L_0,0)\cup(0,L)$. *Then*

$$u_0\in\left[\bar B,0\right)\cup(0,L) \quad\Rightarrow\quad \bar B<u(t)<L,\quad t\in(0,\infty)\,, \tag{41.22}$$

$$u_0\in\left(L_0,\bar B\right) \quad\Rightarrow\quad u_0<u(t),\quad t\in(0,\infty)\,. \tag{41.23}$$

Remark 41.13. According to (41.10), (41.22), (41.23) and Definition 41.4, u is a damped or a homoclinic solution of the auxiliary problem (41.9), (41.2) if and only if u is a damped or a homoclinic solution of the original problem (41.1), (41.2).

Theorem 41.14 (Existence of damped solutions of problem (41.1), (41.2), [2]). *Assume* (41.3)–(41.7), (41.19) *and*

$$\exists\bar B\in(L_0,0):\ \int_0^{\bar B} f(\phi(z))\mathrm{d}z=\int_0^L f(\phi(z))\mathrm{d}z.$$

Then, for each $u_0\in\left[\bar B,L\right)$, *problem* (41.1), (41.2) *has a solution. Every solution of problem* (41.1), (41.2) *with* $u_0\in\left[\bar B,L\right)$ *is damped.*

Example 41.15. Problem (41.1), (41.2) with p, f and ϕ from Example 41.7 has for each $u_0\in[\bar B,L)$ a damped solution.

Remark 41.16. By Theorem 41.14, we can get homoclinic or escape solutions only if $u_0\in\left(L_0,\bar B\right)$.

If $\phi^{-1}\notin\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$, we derive results about the uniqueness by means of Theorems 41.10 and 41.11. Since the next uniqueness result concerns damped solutions, it can be formulated directly for the original problem (41.1), (41.2) due to Remark 41.13.

Theorem 41.17 (Uniqueness of damped solutions, [2]). *Assume* (41.3)–(41.7), (41.16), (41.18), (41.19), (41.20) *and* (41.21). *Let* u *be a damped solution of problem* (41.1), (41.2) *with* $u_0\in(L_0,0)\cup(0,L)$. *Then* u *is a unique solution of this problem.*

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Abstract: *We study analytical properties of a singular nonlinear ordinary differential equation with a ϕ-Laplacian. We investigate solutions of the initial value problem*

$$(p(t)\phi(u'(t)))' + p(t)f(\phi(u(t))) = 0, \quad u(0) = u_0 \in [L_0, L], \quad u'(0) = 0$$

on the half-line $[0,\infty)$. Here, f is a continuous function with three zeros, function p is positive on $(0,\infty)$ and $p(0) = 0$. The integral $\int_0^1 \frac{\mathrm{d}s\,\mathrm{d}s}{p(s)}$ may be divergent which yields the time singularity at $t = 0$. Our equation generalizes equations which appear in hydrodynamics or in the nonlinear field theory.

Keywords: *second order ODE, time singularity, existence and uniqueness, ϕ-Laplacian, damped solution.*

TLUMENÁ ŘEŠENÍ SINGULÁRNÍ ÚLOHY S ϕ-LAPLACIÁNEM

Abstrakt: *Budeme se zabývat chováním řešení singulární obyčejné diferenciální rovnice druhého řádu s ϕ-Laplaciánem*

$$(p(t)\phi(u'(t)))' + p(t)f(\phi(u(t))) = 0$$

na polopřímce $[0,\infty)$ za počátečních podmínek

$$u(0) = u_0 \in [L_0, L], \quad u'(0) = 0.$$

Funkce f je spojitá a má tři nulové body, funkce p je kladná na $(0,\infty)$ a dále platí $p(0) = 0$. Integrál $\int_0^1 \frac{\mathrm{d}s}{p(s)}$ může být divergentní, což způsobuje singularitu v $t = 0$. Naše rovnice zobecňuje rovnice vyskytující se v modelech například v hydrodynamice nebo v nelineární teorii pole.

Klíčová slova: *diferenciální rovnice druhého řádu, časová singularita, existence a jednoznačnost, ϕ-Laplacián, tlumené řešení.*

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